Lecture 22.

Thm 1. Suppose $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then $g(x) = \nu(E_x)$, $h(y) = \mu(E^y)$ are measurable and

$$(\mu \times \nu)(E) = \int \mu(E^y)\, d\nu(y) = \int \nu(E_x)\, d\mu(x).$$

Pf. Note that the conclusion is obvious for rectangles $A \times B$, since $\nu((A\times B)_x) = \nu(B)\chi_A(x)$.

Let $\mathcal{C} = \{E \in \mathcal{M}\otimes\mathcal{N} : \text{conclusion holds}\}$. Then $\mathcal{E} = \{A \times B\} \subseteq \mathcal{C}$. Thus, it suffices to show that $\mathcal{C}$ is a σ-algebra $\Rightarrow \mathcal{M}\otimes\mathcal{N} \subseteq \mathcal{C}$ since $\mathcal{M}\otimes\mathcal{N}$ is the σ-algebra generated by $\mathcal{E}$.

We first show that $\mathcal{C}$ is a <u>monotone class</u>, closed under increasing unions and decreasing intersections.

① Let $E_1 \subseteq E_2 \subseteq \ldots$ s.t. $E_k \in \mathcal{C}$ and $E = \bigcup_{k=1}^{\infty} E_k$. Then, since $E_k \in \mathcal{C}$,

$$(\mu\times\nu)(E) \underset{\text{cont. from below}}{=} \lim_{k\to\infty} (\mu\times\nu)(E_k) = \lim_{k\to\infty} \int \nu((E_k)_x)\, d\mu$$

$$= \lim_{k\to\infty} \int \mu((E_k)^y)\, d\nu$$

Since $E_k \nearrow$, $\nu((E_k)_x) \nearrow$, $\underline{MCT} \Rightarrow$

$$(\mu\times\nu)(E) = \int \lim_{k\to\infty} \nu((E_k)_x)\, d\mu = \int \nu(E_x)\, d\mu$$

and similarly for $\int \mu(E^y)\, d\nu \Rightarrow E \in \mathcal{C}$.

② First, assume μ, ν finite, and let $E_1 \supseteq E_2 \supseteq \dots$, $E_k \in \mathcal{C}$ and $E = \bigcap_{k=1}^{\infty} E_k$.

Since $\mu\times\nu$ is finite, cont. from above $\Rightarrow$

$$(\mu\times\nu)(E) = \lim_{k\to\infty} (\mu\times\nu)(E_k) = \lim_{k\to\infty} \int \nu((E_k)_x)\, d\mu.$$

But $0 \le \nu((E_k)_x) \le \nu(Y) < \infty$ and constants are in $L^1(X,\mu)$ (since $\mu(X) < \infty$),

$\underline{DCT} \Rightarrow$

$$(\mu\times\nu)(E) = \int \lim_{k\to\infty} \nu((E_k)_x)\, d\mu$$

$$= \int \nu(E_x)\, d\mu$$

cont from above again

and similarly for $\int \mu(E^y)\, d\nu$

$\Rightarrow E \in \mathcal{C}$.

If μ, ν are not finite, we find $X_n \nearrow$, $Y_n \nearrow$ with finite measure and $X = \bigcup_{n=1}^{\infty} X_n$, $Y = \bigcup_{n=1}^{\infty} Y_n$. Given E as above, apply previous argument to $E \cap (X_n \times Y_n)$. Details are DIY.

Now, we know $\mathcal{C}$ is a monotone class. It contains $\mathcal{E}$ and the algebra $\mathcal{A}$ of its finite disjoint unions (by linearity), and $\mathcal{C} \subseteq \mathcal{M} \otimes \mathcal{N}$.

The proof is finished by

<u>Monotone Class Lemma</u>. Let $\mathcal{A}$ be an algebra and $\mathcal{C}$ a monotone class s.t. $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{M}(\mathcal{A})$. Then $\mathcal{C} = \mathcal{M}(\mathcal{A})$.

For pf, see Folland 2.35.

$\boxtimes$

Fubini-Tonelli Thm. Let $(X,\mathcal{M},\mu)$, $(Y,\mathcal{N},\nu)$ be σ-finite.

(i) If $f \in L^+(X\times Y)$ then

$$g(x) = \int f(x,y)\,d\nu(y), \quad h(y) = \int f(x,y)\,d\mu(x)$$

are in $L^+(X)$, $L^+(Y)$, resp., and

$$(*) \qquad \int_{X\times Y} f(x,y)\,d(\mu\times\nu) = \int_X \Big(\underbrace{\int_Y f(x,y)\,d\nu(y)}_{g(x)}\Big)\,d\mu(x)$$

$$= \int_Y \Big(\underbrace{\int_X f(x,y)\,d\mu(x)}_{h(y)}\Big)\,d\nu(y).$$

(ii) If $f \in L^1(X\times Y)$, then $f_x \in L^1(Y)$, $f^y \in L^1(X)$ for a.e. x, y. The fcns $g(x)$, $h(y)$ above are in $L^1(X)$, $L^1(Y)$ for a.e. x,y and $(*)$ holds.

Pf. (i) If $f = \chi_E$ for $E \in \mathcal{M} \otimes \mathcal{N}$, then conclusion follows immediately from Thm 1. By linearity, it holds for any simple function in L^+. By Thm 2.10, for any $f \in L^+$ we can find simple $\varphi_k \in L^+$ s.t. $\varphi_k \nearrow f$. Define $g_k(x) = \int \varphi_k(x,y)\, d\nu(y)$, $h_k(y) = \int \varphi_k(x,y)\, d\mu(x)$, which both are in resp. L^+ by above observation (Thm 1 holds for φ_k). Moreover, by MCT,

$$\lim_{k\to\infty} g_k(x) = \int \underbrace{\lim_{k\to\infty} \varphi_k(x,y)}_{f(x,y)}\, d\nu(y) = g(x)$$

$\Rightarrow$ $g \in L^+(X)$ and similarly $h \in L^+(Y)$.

Again by MCT we have

$\longrightarrow$

$$\int_{X\times Y} f(x,y)\,d(\mu\times\nu) = \lim_{k\to\infty} \int_{X\times Y} \varphi_k(x,y)\,d(\mu\times\nu)$$

$$= \lim_{k\to\infty} \int_X g_k(x)\,d\mu(x) = \int g(x)\,d\mu(x)$$

and similarly for $\int h(y)\,d\nu(y)$.

This completes pf of (i).

(ii). Let $f \in L^1(X\times Y)$, and decompose $f = u + iv$, $u = u^+ - u^-$, $v = v^+ - v^-$. Then, $u^\pm, v^\pm \in L^1$. Since f only defined modulo nullsets, so are $u^\pm, v^\pm$ but we can find a representative s.t. $u^\pm, v^\pm \in L^+$. We apply (i) to each one. The corresponding g, h are then in resp. L^1 (since the iterated integrals give full integral). But then $g, h < \infty$ a.e. $\Rightarrow$ f_x, f^y are in resp L^1, as claimed.

By (i), (*) holds for each $u^{\pm}, v^{\pm}$, so by linearity for f as claimed. ☒